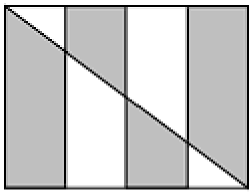
# October Math Gems

## Problem of the week 27

## §1 Problems

**Problem 1.1.**

Below is a diagram showing a $6 \times 8$ rectangle divided into four $6 \times 2$ rectangles and one diagonal line. Find the total perimeter of the four shaded trapezoids.



*Solution.* $\boxed{\textbf{Answer: 48}}$

The vertical sides of the shaded trapezoids are formed by all 5 of the vertical line segments, each of which has length 6. Together, the horizontal sides of the shaded trapezoids make one horizontal side of the rectangle of length 8. The other sides of the trapezoids are formed by the diagonal of the rectangle which has length given by the Pythagorean Theorem to be $\sqrt{6^2 + 8^2} = 10$. The total of the perimeters is,therefore, $5 \cdot 6 + 8 + 10 = 48$. □

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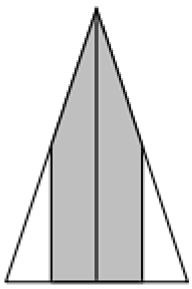
**Problem 1.2.**

An isosceles triangle has a base with length 12 and the altitude to the base has length 18. Find the area of the region of points inside the triangle that are a distance of at most 3 from that altitude.



*Solution.* $\boxed{\textbf{Answer: 81}}$

The region of points inside the triangle that are greater than 3 from the altitude (the unshaded region in the diagram) together form a triangle that is similar to the original triangle but with half the dimensions. Thus, its area is $\frac{1}{4}$ the area of the original triangle, and the requested area is $\frac{3}{4}$ the area of the original triangle. The original triangle has area $\frac{1}{2} \cdot 12 \cdot 18 = 108$. The requested area is then $\frac{3}{4} \cdot 108 = 81$.

□

**Problem 1.3.**

Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

*Solution.* $\boxed{\textbf{Answer: 50}}$
Of the 450 students originally at the concert, $(0.40) \cdot 450 = 180$ were boys. Let $n$ be the number of boys on the bus, so the number of students on the bus was $2n$. Then $0.41 = \frac{180+n}{450+2n}$, which simplifies to $(0.41) \cdot 450 + 0.82n = 180 + n$ and

$$n = \frac{0.41 \cdot 450 - 180}{1 - 0.82} = \frac{4.5}{0.18} = 25.$$

Thus, the number of students on the bus was $2 \cdot 25 = 50$.

□

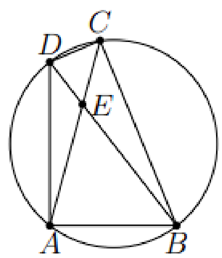
**Problem 1.4.**

In quadrilateral $ABCD$, let $AB = 7$, $BC = 11$, $CD = 3$, $DA = 9$, $\angle BAD = \angle BCD = 90°$, and diagonals $\overline{AC}$ and $\overline{BD}$ intersect at $E$. The ratio $\frac{BE}{DE} = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Solution.* $\boxed{\textbf{Answer: 104}}$
Because $\measuredangle BAD = \measuredangle BCD = 90°$,quadrilateral $ABCE$ is cyclic. Thus $\measuredangle BDC = \measuredangle BAC$ and $\measuredangle ACD = \measuredangle ABD$, so $\Delta EAB \sim \Delta EDC$. Similarly, $\Delta EDA \sim \Delta ECB$. Therefore,

$$\frac{BE}{DE} = \frac{BE}{BA} \cdot \frac{BA}{DE} = \frac{CE}{CD} \cdot \frac{BA}{DE} = \frac{BA}{CD} \cdot \frac{CE}{DE} = \frac{BA}{CD} \cdot \frac{CB}{DA} = \frac{7}{3} \cdot \frac{11}{9} = \frac{77}{27}$$

The requested sum is 77+27=104.




□

**Problem 1.5.**

Let $a$ be a real number such that

$$5\sin^4\left(\frac{a}{2}\right) + 12\cos a = 5\cos^4\left(\frac{a}{2}\right) + 12\sin a.$$

There are relatively prime positive integers $m$ and $n$ such that $\tan a = \frac{m}{n}$. Find $10m + n$.

*Solution.* $\boxed{\textbf{Answer: 82}}$
Rewrite the given equation as

$$12\,cos\,(a) - 12\,sin\,(a) = 5\,cos^4\left(\frac{a}{2}\right) - 5\,sin^4\left(\frac{a}{2}\right).$$

Then

$$12\,cos\,(a) - 12\,sin\,(a) = 5\left(cos^2\left(\frac{a}{2}\right) - 5\,sin^2\left(\frac{a}{2}\right)\right)\left(cos^2\left(\frac{a}{2}\right) + 5\,sin^2\left(\frac{a}{2}\right)\right)$$

After applying the following trig identities

$$sin^2\theta + cos^2\theta = 1 \quad ,\text{and} \quad sin^2\theta - cos^2\theta = cos\,2\theta$$

Then

$$\begin{aligned} 12\,cos\,(a) - 12\,sin\,(a) &= 5\,cos\,(a) \\ 7\,cos\,(a) &= 12\,sin\,(a) \\ tan\,(a) &= \frac{sin\,(a)}{cos\,(a)} = \frac{7}{12} \end{aligned}$$

It follows that $tan\,a = \frac{7}{12}$. The requested expression is $10 \cdot 7 + 12 = 82$. The equation is satisfied by $a$ is approximately equal to 0.5281 (in radians) or 30.26 (in degrees).

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**Problem 1.6.** Let $x$ be a real number such that $\left(\sqrt{6}\right)^x - 3^x = 2^{x-2}$. Evaluate $\dfrac{4^{x+1}}{9^{x-1}}$.

*Solution.* $\boxed{\textbf{Answer: 576}}$
Let $u = (\sqrt{2})^x$ and $v = (\sqrt{3})^x$. Then $uv - v^2 = \frac{u^2}{4}$, implying $\left(\dfrac{u}{2} - v\right)^2 = 0$. Then $\dfrac{u}{v} = 2$, so $\left(\dfrac{4}{9}\right)^x = 2^4$.

The requested expression is $\dfrac{4^{x+1}}{9^{x-1}} = 4 \cdot 9 \cdot \left(\dfrac{4}{9}\right)^x = 36 \cdot 16 = 576.$

□

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**Problem 1.7.**

Let $ABCD$ be a convex quadrilateral inscribed in a circle with $AC = 7$, $AB = 3$, $CD = 5$, and $AD - BC = 3$. Then $BD = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Solution.* $\boxed{\textbf{Answer: 62}}$
Let $BC = x$. Because $\measuredangle B$ and $\measuredangle D$ are opposite angles in a cyclic quadrilateral, they are supplementary, so $cos\,B + cos\,D = 0$. Apply the Law of Cosines to $\Delta ABC$ and $\Delta ADC$ to find

$$cos\,B + cos\,D = \frac{3^2 + x^2 - 7^2}{2 \cdot 3 \cdot x} + \frac{5^2 + (x+3)^2 - 7^2}{2 \cdot 5 \cdot (x+3)} = 0.$$

It follows that $5(x+3)(x^2 - 40) + 3x(x^2 + 6x - 15) = 0$, which reduces to $8x^3 + 33x^2 - 245x - 600 = (x-5)(8x^2 + 73x + 120) = 0$. Notice that $x = 5$ is the only solution as $8x^2 + 73x + 120$ is always positive when $x$ is positive. Hence, $BC = 5$, $AD = 8$, and, from Ptolemy's Theorem, $7 \cdot BD = 3 \cdot 5 + 5 \cdot 8$, implying $BD = \frac{55}{7}$. The requested sum is 55+7=62

□

**Problem 1.8.**

The sum of the solutions to the equation

$$x^{\log_2 x} = \frac{64}{x}$$

can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

*Solution.* $\boxed{\textbf{Answer: 41}}$

Taking the logarithm base 2 of both sides of the equation gives

$$log_2\left(x^{log_2 x}\right) = log_2\left(\frac{64}{x}\right)$$

After applying the following logarithmic identities

$$log(a^b) = b \cdot log(a) \quad ,and \quad log(\frac{a}{b}) = log(a) - log(b)$$

we get $(log_2(x))(log_2(x)) = 6 - log_2(x)$ ,and so $(log_2(x))^2 + log_2(x) - 6 = 0$. This is equivalent to $(log_2(x) - 2)(log_2(x) + 3) = 0$, whose solutions are 4 and $\frac{1}{8}$.
The sum of the solutions is $\frac{33}{8}$. The requested sum is $33 + 8 = 41$

□

**Problem 1.9.**

Find the maximum possible value obtainable by inserting a single set of parentheses into the expression $1 + 2 \times 3 + 4 \times 5 + 6$.

*Solution.* $\boxed{\textbf{Answer: 77}}$

The maximum possible value is obtained with the expression
$1 + 2 \times (3 + 4) \times 5 + 6 = 77$

□

**Problem 1.10.**

Let $a$ be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

*Solution.* $\boxed{\textbf{Answer: 1265}}$

Adding 4 to each side of the given equation yields

$$4a^2 + 4 + \frac{1}{a^2} = 121 = 11^2$$
$$(2a + \frac{1}{a})^2 = 11^2$$

So

$$2a + \frac{1}{a} = 11$$

Now, cube both sides yields

$$\begin{aligned} 11^3 &= 8a^3 + 3 \cdot 4a + 3 \cdot \frac{2}{a} + \frac{1}{a^3} \\ &= 8a^3 + \frac{1}{a^3} + 6\left(2a + \frac{1}{a}\right) \\ &= 8a^3 + \frac{1}{a^3} + 6 \cdot 11 \end{aligned}$$

It follows that

$$8a^3 + \frac{1}{a^3} = 11^3 - 6 \cdot 11 = 1331 - 66 = 1265$$

□

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**Problem 1.11.**

In $\triangle ABC$ let point $D$ be the foot of the altitude from $A$ to $\overline{BC}$. Suppose that $\angle A = 90^\circ$, $AB - AC = 5$, and $BD - CD = 7$. Find the area of $\triangle ABC$.

*Solution.* $\boxed{\textbf{Answer: 150}}$

Let $AC = x$ and $AD = h$. Then $AB^2 = (x+5)^2 = BD^2 + h^2$ and $AC^2 = x^2 = CD^2 + h^2$ , implying

$$(x+5)^2 - x^2 = (BD - CD)(BD + CD).$$

Thus, $10x + 25 = 7BC$ and, form the Pythagorean Theorem,

$$\frac{(10x+25)^2}{49} = (x+5)^2 + x^2.$$

This equation reduces to $x^2 + 5x - 300 = 0$, whose positive solution is $x = 15$. Hence, $AC = 15$, $AB = 20$, and $BC = 25$, implying $AD = h = 12$. The requested area is $\dfrac{AD \cdot BC}{2} = \dfrac{12 \cdot 25}{2} = 150$.

□

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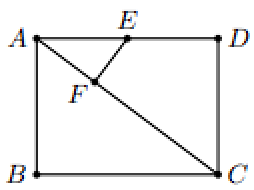
**Problem 1.12.**

A rectangle with width 30 inches has the property that all points in the rectangle are within 12 inches of at least one of the diagonals of the rectangle. Find the maximum possible length for the rectangle in inches.

*Solution.* $\boxed{\textbf{Answer: 40}}$

Label the rectangle $ABCD$ with $AB = 30$. Let $E$ be the midpoint of $\overline{AD}$, and $F$ be the perpendicular projection of $E$ onto the diagonal $\overline{AC}$. Let $x = AE$.



Because the midpoints of the sides of a rectangle are the points on the rectangle farthest from the diagonals, $x$ is as great as possible when $EF = 12$. Because $\Delta ACD \simeq \Delta AEF$,

$$\frac{AC}{CD} = \frac{AE}{EF} \quad so \quad \frac{\sqrt{30^2 + 4x^2}}{30} = \frac{x}{12},$$

which simplifies to $x = 20$. Thus, the requested side length is $AD = 2x = 40$ inches. Note that on any rectangle, the four midpoints of the sides are all the same distance from the diagonals of the rectangle

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**Problem 1.13.**

Let $a$ and $b$ be positive integers satisfying $3a < b$ and $a^2 + ab + b^2 = (b + 3)^2 + 27$. Find the minimum possible value of $a + b$.

*Solution.* $\boxed{\textbf{Answer: 25}}$
Squaring the binomial gives $a^2 + ab + b^2 = b^2 + 6b + 9 + 27$ which simplifies to $0 = a^2 + ab - 6b - 36 = (a - 6)(a + b + 6)$. Because $a$ and $b$ are positive, $a + b + 6 > 0$, so it must be that $a = 6$. Because $3a < b$, the least value of $b$ that satisfies the given conditions is $b = 19$. The minimum possible value of $a + b$ is $6 + 19 = 25$

□

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**Problem 1.14.**

Find the positive integer $n$ such that a convex polygon with $3n + 2$ sides has 61.5 percent fewer diagonals than a convex polygon with $5n - 2$ sides.

*Solution.* $\boxed{\textbf{Answer: 26}}$
A convex polygon with $k$ sides has $\dfrac{k(k-3)}{2}$ diagonals, so

$$\frac{(3n+2)(3n-1)}{2} = (1 - 0.615)\frac{(5n-2)(5n-5)}{2}.$$

It follows that $(3n+2)(3n-1) = \dfrac{77}{200}(5n-2)\cdot 5(n-1)$, implying that $40(9n^2+3n-2) = 77(5n^2-7n+2)$. This reduces to $25n^2-659n+234 = 0$, which has integer solution $n = 26$

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**Problem 1.15.** Find the number of divisors of $20^{22}$ that are perfect squares.

*Solution.* $\boxed{\textbf{Answer: 276}}$
A perfect squares that divides $20^22 = 2^44 \times 5^22$ is of the form $2^{2m} \times 5^{2n}$, where $m$ is an integer with $0 \le m \le 22$ and $n$ is an integer with $0 \le n \le 11$. Hence, the number of perfect squares divisors is $(22 + 1)(11 + 1) = 276$.

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### Problem 1.16.

The value of

$$\left(1-\frac{1}{2^2-1}\right)\left(1-\frac{1}{2^3-1}\right)\left(1-\frac{1}{2^4-1}\right)\cdots\left(1-\frac{1}{2^{29}-1}\right)$$

can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $2m-n$.

*Solution.* $\boxed{\textbf{Answer: 1}}$
Note that for any positive integer $k$

$$1-\frac{1}{2^k-1}=\frac{2^k-2}{2^k-1}=2\cdot\frac{2^{k-1}-1}{2^k-1}.$$

Hence, the given product is equal to

$$\left(2\cdot\frac{2-1}{2^2-1}\right)\left(2\cdot\frac{2^2-1}{2^3-1}\right)\left(2\cdot\frac{2^3-1}{2^4-1}\right)\cdots\left(2\cdot\frac{2^{28}-1}{2^{29}-1}\right)$$

which telescopes to $\dfrac{2^{28}}{2^{29}-1}$. The requested expression is $2\cdot 2^{28}-(2^{29}-1)=1$.

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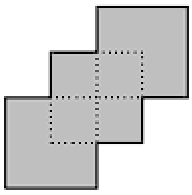
### Problem 1.17.

The 12-sided polygon below was created by placing three $3\times 3$ squares with their sides parallel so that vertices of two of the squares are at the center of the third square. Find the perimeter of this 12-sided polygon.



*Solution.* $\boxed{\textbf{Answer: 24}}$
The perimeter is the same as the sum of the perimeters of two of the squares which is $2\cdot 4\cdot 3=24$.

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### Problem 1.18.

A rectangular wooden block has a square top and bottom, its volume is 576, and the surface area of its vertical sides is 384. Find the sum of the lengths of all twelve of the edges of the block.

*Solution.* $\boxed{\textbf{Answer: 112}}$
Suppose the side length of the square top of the block is $s$, and the height of the block is $h$. Then the volume is $576=hs^2$ and the surface area if the vertical sides is $384=4hs$. Dividing the first equation by the second gives

$$\frac{576}{384}=\frac{hs^2}{4hs}.$$

which simplifies to $s = 6$. Then $384 = 4 \cdot 6 \cdot h$, so $h = 16$. The sum of the lengths of the edges of the box is $8s + 4h = 8 \cdot 6 + 4 \cdot 16 = 112$

$\square$

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**Problem 1.19.**

Find the value of $x$ where the graph of

$$y = \log_3\left(\sqrt{x^2 + 729} + x\right) - 2\log_3\left(\sqrt{x^2 + 729} - x\right)$$

crosses the $x$-axis.

*Solution.* $\boxed{\textbf{Answer: 36}}$

Because $\left(\sqrt{x^2 + 729} + x\right)\left(\sqrt{x^2 + 729} - x\right) = 729$, when $y = 0$,

$$log_3\left(\sqrt{x^2 + 729} + x\right) = log_3\left(\frac{729}{\sqrt{x^2 + 729} + x}\right)^2.$$

This implies that $\left(\sqrt{x^2 + 729} + x\right)^3 = 729^2$ from which it follows that $\sqrt{x^2 + 729} + x = 9^2$. Thus, $\sqrt{x^2 + 729} = 81 - x$, so $x^2 + 729 = 6561 - 162x + x^2$ and

$$x = \frac{6561 - 729}{162} = 36.$$

$\square$

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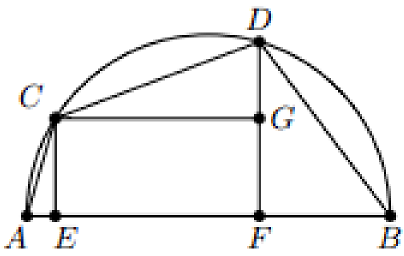
**Problem 1.20.**

A semicircle has diameter $\overline{AB}$ with $AB = 100$. Points $C$ and $D$ lie on the semicircle such that $AC = 28$ and $BD = 60$. Find $CD$.

*Solution.* $\boxed{\textbf{Answer: 60}}$

Let $E$ and $F$ be the projections of $C$ and $D$, respectively, onto $\overline{AB}$, and let $G$ be the projection of $C$ onto $\overline{DF}$, as shown




Because $\measuredangle ACB = 90°$, it follows that $\Delta ACB \sim \Delta AEC \sim CEB$. The Pythagorean Theorem gives $BC = \sqrt{100^2 - 28^2} = 96$, so $AE = 28 \cdot \frac{28}{100}$ and $CE = 96 \cdot \frac{28}{100}$. Similarly, $AD = \sqrt{100^2 - 60^2} = 80$, so $BF = 60 \cdot \frac{60}{100} = 36$ and $DF = 80 \cdot \frac{60}{100} = 48$. Thus,

$DG = DF - GF = DF - CE = 48 - \frac{96 \cdot 28}{100}$ and $CG = EF = 100 - AE - BF = 100 - 28 \cdot \frac{28}{100} - 36 = 64 - \frac{28^2}{100}$. Then The Pythagorean Theorem gives

$$CD = \sqrt{DG^2 + CG^2} = \sqrt{\left(64 - \frac{28^2}{100}\right)^2 + \left(48 - \frac{96 \cdot 28}{100}\right)^2}$$
$$= \frac{16}{100}\sqrt{(400 - 49)^2 + (300 - 168)^2} = \frac{4}{25}\sqrt{9(117^2 + 44^2)} = \frac{12}{25} \cdot 125 = 60.$$

Alternatively, one can use Ptolemy's Theorem applied to cyclic quadrilateral $ABDC$ to get

$$CD = \frac{AC \cdot BD - AD \cdot BC}{AB} = \frac{80 \cdot 96 - 28 \cdot 60}{100} = 60.$$

One can also use trigonometry. Let $\theta = \measuredangle CAD = \measuredangle CBD$. As above, the Pythagorean Theorem gives $AD = 80$ and $BC = 96$, so the law of Cosines applied to $\Delta CAD$ and to $\Delta CBD$ gives

$$CD^2 + 28^2 + 80^2 - 2 \cdot 28 \cdot 80 \cdot cos\theta = 96^2 + 60^2 - 2 \cdot 96 \cdot 80 \cdot cod\theta,$$

From which one gets that $cos\theta = \frac{4}{5}$. Then $CD^2 = 3600$ and $CD = 60$, as above $\square$